# Lifetime effect on the superfluidity in neutron stars



1974 J. Phys. A: Math. Nucl. Gen. 7 1970

(http://iopscience.iop.org/0301-0015/7/15/023)

J. Phys. A: Math., Nucl. Gen., Vol. 7, No. 15, 1974. Printed in Great Britain. 

# Lifetime effect on the superfluidity in neutron stars

Naoki Itoh and M A Alpar
Cavendish Laboratory, University of Cambridge, Madingley Road, Cambridge CB3 0HE, UK

Received 25 February 1974

**Abstract.** A true lifetime effect on the superfluidity in neutron stars due to quasi-particle scattering processes is calculated with the Green function method and is found to lower the superfluid transition temperature by about 20%. This effect constitutes part of the impurity effects on the superfluidity in neutron stars, ie it partly arises from the presence of neutrons in the proton superfluid phase and protons in the neutron superfluid phase.

## 1. Introduction

The superfluidity of the neutron and proton liquids has been one of the most interesting problems of the physics of neutron stars. Superfluidity will affect the cooling of neutron stars (Itoh 1969, Itoh and Tsuneto 1972, Tsuruta *et al* 1972), and the relaxation time after neutron star quakes (Baym *et al* 1969). Many calculations have been done for the evaluation of the transition temperatures of the nucleon superfluid (see Takatsuka 1972 for detailed references). However, to the best of the authors' knowledge, no calculation has been made using the fully renormalized interaction between nucleons. Furthermore, in calculating the superfluid transition temperature of the proton (neutron) liquid, one should take into account the effect of the neutron (proton) liquid. This impurity effect can be incorporated in the renormalization procedure. As a first step towards a fully renormalized theory, we shall investigate in this paper the effect of the lifetime due to quasi-particle scattering on superfluidity in neutron stars. This effect is negligible in ordinary superconductors, but there is no reason why this should be the case for superfluids in neutron stars. In fact it will be found that the lifetime effect lowers the proton superfluid transition temperature by about 20%. Our approach will be analogous to that of Abrikosov and Gor'kov (1960) in their treatment of the effect of magnetic impurities on the critical temperatures of superconductors. The calculation will be based on the Green function method by using the notation of Nambu (1960) in order to emphasize the time-reversal symmetry.

In § 2 the self-energy equation will be derived by taking quasi-particle scattering into account. In § 3 the lowering of the superfluid transition temperature will be evaluated numerically. Section 4 will be devoted to the discussion of results and the conclusion.

## 2. The self-energy equation

Abrikosov and Gor'kov (1960) have shown how magnetic impurities reduce the transition temperature of a superconductor: this results from the true lifetime effect caused by the magnetic interaction which breaks time-reversal symmetry. For the superfluid in a neutron star, the most important process giving rise to a lifetime will be quasi-particle scattering. Since neutron and proton liquids co-exist in the neutron star, scattering could involve nucleons of the different kind as well as the same kind. Let us consider the quasi-particle scattering of different nucleons, and assume that the scatterer (say a neutron) is in a normal state. This assumption means that the temperature considered is higher than the superfluid transition temperature of the scatterer nucleon liquid. In the case of neutron stars this range of temperatures $T_{c,\mathrm{neutron}} < T < T_{c,\mathrm{proton}}$ is quite wide, because according to Takatsuka (1972) the high-density neutron liquid leads to a $^3P_2$ pairing for the neutron superfluid for which the transition temperature is an order of magnitude lower than that of the low-density proton superfluid with $^1S_0$ pairing. The diagram considered here is shown in figure 1.

The renormalized Green function of the nucleon in the presence of the impurity liquid is given by

$$G_\omega(\boldsymbol{p}) = (\mathrm{i}\tilde{\omega} - \xi\rho_3 - \tilde{\Delta}\rho_1\sigma_2)^{-1} \tag{1}$$

where $\tilde{\omega}$ and $\tilde{\Delta}$ are the renormalized frequency and order parameter, respectively. We have introduced the matrix notation of Nambu (1960). The symbols $\sigma_i$ and $\rho_i$ are Pauli matrices operating on the spin space and the space composed of particle and hole states, respectively. We have to solve the Dyson equation for the case of finite temperature:

$$G_\omega^{-1}(\boldsymbol{p}) = (G_\omega^0(\boldsymbol{p}))^{-1} - \Sigma_\omega(\boldsymbol{p}) \tag{2}$$

$$G_\omega^0(\boldsymbol{p}) = (\mathrm{i}\omega - \xi\rho_3 - \Delta\rho_1\sigma_2)^{-1}. \tag{3}$$

In the Born approximation, we obtain

$$\begin{aligned}\Sigma_\omega(\boldsymbol{p}) = {} & \frac{1}{4(2\pi)^6}\int\!\!\int \mathrm{d}\boldsymbol{q}\,\mathrm{d}\boldsymbol{k}(U(\boldsymbol{q}))^2\left(\tanh\frac{\xi_{k+q}}{2T} - \tanh\frac{\xi_k}{2T}\right)\\ & \times\left[\left(1 + \frac{\xi_{p-q}\rho_3 + \tilde{\Delta}\rho_1\sigma_2}{(\xi_{p-q}^2 + \tilde{\Delta}^2)^{1/2}}\right)\left(\coth\frac{\xi_{k+q}-\xi_k}{2T} + \tanh\frac{(\xi_{p-q}^2+\tilde{\Delta}^2)^{1/2}}{2T}\right)\right.\\ & \times\left(\frac{1}{\mathrm{i}\tilde{\omega} - (\xi_{p-q}^2+\tilde{\Delta}^2)^{1/2} - (\xi_{k+q}-\xi_k)}\right)\\ & + \left(1 - \frac{\xi_{p-q}\rho_3 + \tilde{\Delta}\rho_1\sigma_2}{(\xi_{p-q}^2 + \tilde{\Delta}^2)^{1/2}}\right)\left(\coth\frac{\xi_{k+q}-\xi_k}{2T} - \tanh\frac{(\xi_{p-q}^2+\tilde{\Delta}^2)^{1/2}}{2T}\right)\\ & \left.\times\left(\frac{1}{\mathrm{i}\tilde{\omega} + (\xi_{p-q}^2+\tilde{\Delta}^2)^{1/2} - (\xi_{k+q}-\xi_k)}\right)\right]\end{aligned} \tag{4}$$

where $T(\equiv\beta^{-1})$ is the temperature and $U(\boldsymbol{q})$ is the Fourier transform of the pseudo-potential. Now we carry out the analytical continuation to real frequencies. Neglecting the real part of the self-energy, we have the following equations for the renormalization of the diagonal and the off-diagonal parts of the self-energy:

$$\begin{aligned}\tilde{\omega} = {} & \omega + \mathrm{i}\left(\frac{\Gamma_1}{\Omega_1^2} + \frac{\Gamma_2}{\Omega_2^2}\right)2(\mathrm{e}^{\beta\tilde{\omega}}+1)\int_{\tilde{\Delta}}^{\infty}\mathrm{d}E_{q'}\frac{E_{q'}}{(E_{q'}^2 - \tilde{\Delta}^2)^{1/2}}\\ & \times\left(\frac{\tilde{\omega} - E_{q'}}{\{\exp[\beta(\tilde{\omega} - E_{q'})] - 1\}[\exp(\beta E_{q'}) + 1]} + \frac{\tilde{\omega} + E_{q'}}{\{\exp[\beta(\tilde{\omega} + E_{q'})] - 1\}[\exp(-\beta E_{q'}) + 1]}\right)\end{aligned} \tag{5}$$

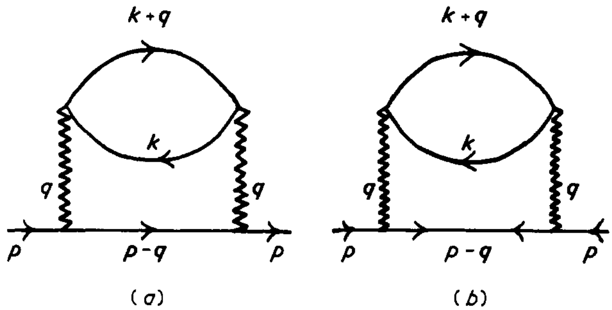


**Figure 1.** (*a*) The diagonal part of the self-energy diagram corresponding to the quasi-particle scattering. (*b*) The off-diagonal part of the self-energy diagram.

$$\tilde{\Delta} = \Delta + \mathrm{i}\left(\frac{\Gamma_1}{\Omega_1^2} - \frac{\Gamma_2}{\Omega_2^2}\right) 2(\mathrm{e}^{\beta\tilde{\omega}} + 1) \int_{\tilde{\Delta}}^{\infty} \mathrm{d}E_{q'} \frac{\tilde{\Delta}}{(E_{q'}^2 - \tilde{\Delta}^2)^{1/2}}$$
$$\times \left( \frac{\tilde{\omega} - E_{q'}}{\{\exp[\beta(\tilde{\omega} - E_{q'})] - 1\}[\exp(\beta E_{q'}) + 1]} - \frac{\tilde{\omega} + E_{q'}}{\{\exp[\beta(\tilde{\omega} + E_{q'})] - 1\}[\exp(-\beta E_{q'}) + 1]} \right). \tag{6}$$

Here $\Gamma_1/\Omega_1^2$ and $\Gamma_2/\Omega_2^2$ correspond to non-spin-flip and spin-flip scattering, respectively. Let us remark on the sign reversal for $\Gamma_2/\Omega_2^2$. Since it is very difficult to handle equations (5) and (6) for general values of $\tilde{\Delta}$, contrasting with the case of static impurities which Abrikosov and Gor'kov (1960) have investigated, we shall investigate the limiting case $\tilde{\Delta} \to 0$. In this limit, (5) and (6) reduce to

$$\tilde{\omega} = \omega + \mathrm{i}\left(\frac{\Gamma_1}{\Omega_1^2} + \frac{\Gamma_2}{\Omega_2^2}\right)(\tilde{\omega}^2 + \pi^2 T^2) \tag{7}$$

$$\tilde{\Delta} = \Delta + \mathrm{i}\left(\frac{\Gamma_1}{\Omega_1^2} - \frac{\Gamma_2}{\Omega_2^2}\right) 4\pi\tilde{\Delta}\tilde{\omega}\left(\int_0^{\infty} \mathrm{d}x \frac{\mathrm{e}^{\pi x} - \cos(\beta\tilde{\omega}x)}{\mathrm{e}^{2\pi x} - 1} - \frac{1}{2\pi}\right). \tag{8}$$

Equation (7) shows the diagonal part of the self-energy which is well known in the theory of Fermi liquids. The integral in equation (8) has been evaluated numerically and plotted in figure 2. The second large bracket in equation (8) is positive definite.

## 3. Lowering of the superfluid transition temperature

We use the BCS model in order to see how the lifetime due to quasi-particle scattering affects the superfluid transition temperature. In the BCS model, the order parameter $\Delta$ at temperature $T$ is determined by

$$\Delta = \frac{N(0)V}{2} \int_{-\omega_{D'}}^{\omega_{D'}} \mathrm{d}\omega \, \mathrm{Re} \frac{\tilde{\Delta}}{(\omega^2 - \tilde{\Delta}^2)^{1/2}} \tanh \tfrac{1}{2}\beta\omega, \tag{9}$$

$$\omega_{D'} = (\omega_D^2 + \Delta^2)^{1/2} \tag{10}$$

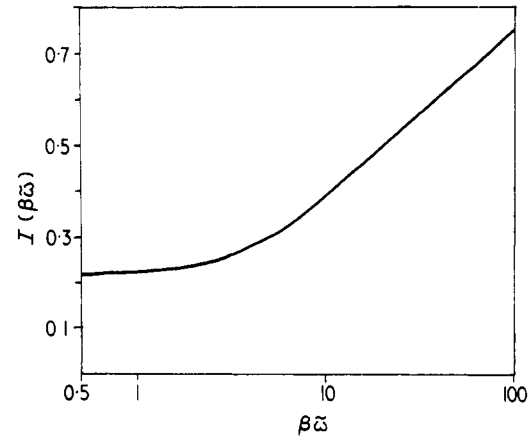


**Figure 2.** $I(\beta\tilde{\omega}) = \int_0^\infty dx \frac{e^{\pi x} - \cos(\beta\tilde{\omega}x)}{e^{2\pi x} - 1}$.

$\omega_D$ being the BCS cut-off. Now let us introduce the parameter

$$u = \tilde{\omega}/\tilde{\Delta}. \tag{11}$$

The critical temperature $T_c$, at which the order parameter $\Delta$ tends to zero is expressed by

$$1 = N(0)V\int_0^{\omega_D} d\omega \, \mathrm{Re}\frac{1}{u\Delta} \tanh \tfrac{1}{2}\beta_c\omega. \tag{12}$$

On the other hand, from equations (7), (8) and (11) we obtain

$$u\Delta = \omega + i\frac{\Gamma_1}{\Omega_1^2}\left\{\pi^2T^2 + \tilde{\omega}^2\left[1 - 4\pi\left(I(\beta\tilde{\omega}) - \frac{1}{2\pi}\right)\right]\right\}$$
$$+ i\frac{\Gamma_2}{\Omega_1^2}\left\{\pi^2T^2 + \tilde{\omega}^2\left[1 + 4\pi\left(I(\beta\tilde{\omega}) - \frac{1}{2\pi}\right)\right]\right\} \equiv \omega + i\epsilon \tag{13}$$

where

$$I(\beta\tilde{\omega}) = \int_0^\infty dx \frac{e^{\pi x} - \cos(\beta\tilde{\omega}x)}{e^{2\pi x} - 1}. \tag{14}$$

Equation (12) is reduced to

$$1 = N(0)V\int_0^{\omega_D} d\omega \frac{\omega}{\omega^2 + \epsilon^2} \tanh \tfrac{1}{2}\beta_c\omega. \tag{15}$$

The coefficient of $\tilde{\omega}^2$ for the non-spin-flip scattering in equation (13) becomes negative for $\beta\tilde{\omega} > 2{\cdot}14$, while that for the spin-flip scattering is always positive. Thus the non-spin-flip scattering and the spin-flip scattering contribute to the true lifetime in different ways. This is to some extent analogous to the case of static impurities (Abrikosov and Gor'kov 1960); however, the contribution from the non-spin-flip scattering is nonzero in our case. When the coefficient of $\tilde{\omega}^2$ in equation (13) is positive, neglecting the $\tilde{\omega}^2$ dependence of $\epsilon$ will lead to an underestimate of the lowering of the transition temperature. For $\beta\tilde{\omega} \lesssim 10$, the coefficient will be positive if

$$\left(\frac{\Gamma_1}{\Omega_1^2}\right)\left(\frac{\Omega_2^2}{\Gamma_2}\right) \lesssim 2.$$

On assuming this to be the case and neglecting the $\tilde{\omega}^2$ term, $\epsilon$ becomes

$$\epsilon = \left(\frac{\Gamma_1}{\Omega_1^2}+\frac{\Gamma_2}{\Omega_2^2}\right)\pi^2T^2 \equiv \frac{\Gamma}{\Omega^2}\pi^2T^2. \tag{16}$$

Inserting this into (15), and making use of the identity

$$(N(0)V)^{-1} = \ln\left(\frac{2\gamma\omega_D}{\pi T_{c0}}\right); \qquad \gamma = 1{\cdot}78 \tag{17}$$

where $T_{c0}$ is the superfluid transition temperature in the absence of quasi-particle scatterings, we obtain

$$\ln\left(\frac{T_c}{T_{c0}}\right) = \psi(\tfrac{1}{2})-\psi(\tfrac{1}{2}+\rho_c) \tag{18}$$

$$\rho_c = \frac{\Gamma}{\Omega^2}\frac{\pi}{2}T_c \tag{19}$$

where $\psi(x)$ is the di-gamma function. The solution has been calculated numerically and is shown in figure 3. We note that the transition temperature is always finite for finite values of $\alpha$, in contrast with the case of static impurities where the transition temperature vanishes for a critical strength of the magnetic interaction (Abrikosov and Gor'kov 1960). This is due to the temperature-squared dependence of the lifetime. For small values of $\alpha = \frac{1}{2}\pi(\Gamma/\Omega^2)T_{c0}$, the solution of equation (18) is approximated by

$$\frac{T_c}{T_{c0}} = 1-\tfrac{1}{2}\pi^2\alpha. \tag{20}$$

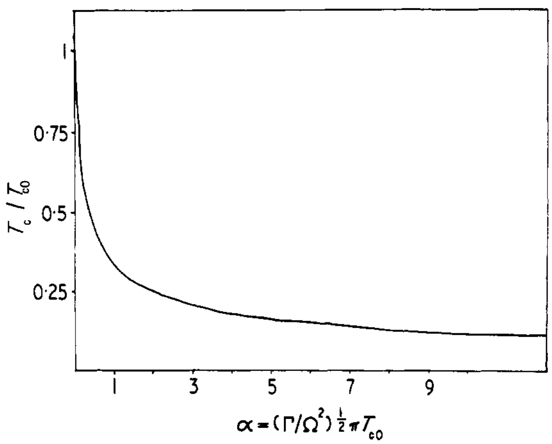


**Figure 3.** Lowering of the transtion temperature as a function of

$$\alpha \equiv \frac{\pi}{2}\frac{\Gamma}{\Omega^2}T_{c0}.$$

We can now apply equation (18) to evaluate the effect of scattering by neutron quasi-particles on the proton superfluid transition temperature, assuming that the superfluid transition temperature for neutrons is lower than that for protons, as discussed in § 2. We shall express $\Gamma/\Omega^2$ in terms of an s-wave scattering amplitude $a$:

$$\frac{\Gamma}{\Omega^2} = \frac{2}{\pi} \frac{m_p^* a^2}{\hbar^2} \tag{21}$$

where $m_p^*$ is the effective mass of the proton. The s-wave scattering amplitude $a$ will be determined so as to fit the differential cross section for proton–neutron scattering at energies of interest. For $T_{c0} = 1{\cdot}5$ MeV, we have $\alpha \sim 0{\cdot}02$ corresponding to the typical condition in neutron stars. From figure 3, we see that we have 8·5 % lowering of the proton superfluid transition temperature.

In addition to the quasi-particle scattering by neutron quasi-particles, there is also a term due to scattering by other proton quasi-particles. This process will give rise to $\alpha \sim 0{\cdot}03$ corresponding to the typical conditions of $^1S_0$ proton superfluidity in neutron stars. In total, we predict a twenty per cent depression of the proton superfluid transition temperature due to the lifetime effect. The neutron superfluid also suffers a lowering of the transition temperature because of the similar lifetime effect due to quasi-particle scattering.

## 4. Conclusion

We have made use of the BCS model to make the problem solvable. However, the BCS model is insufficient in describing the superfluidity of the nucleon liquid. Therefore, the numerical values in the preceding section should not be taken too seriously. Nevertheless it seems clear that the lifetime effect due to quasi-particle scattering lowers the superfluid transition temperature appreciably.

It has also been shown that the impurity effect of the neutron liquid on the proton superfluid is not so drastic as in the analogous case of static impurities (Abrikosov and Gor'kov 1960), because neutrons are strongly degenerate at low temperatures.

We have considered only quasi-particle scattering in order to renormalize the interaction between nucleons. However, there exists another very important effect which should be taken into account, ie, the polarization of the medium (Pines 1971). It is hoped that further investigations will be carried out by taking both effects into account.

## Acknowledgments

The authors wish to thank Professor P W Anderson for his critical comments. One of the authors (NI) wishes to thank Professor D Pines for his hospitality at University of Illinois where this work was initiated.

## References

Abrikosov A A and Gor'kov L P 1960 *Zh. Eksp. Teor. Fiz.* **39** 1781 (1960 *Sov. Phys.–JETP* **12** 1243)
Baym G, Pethick C, Pines D and Ruderman M 1969 *Nature, Lond.* **224** 872

Itoh N 1969 *Prog. Theor. Phys.* **42** 1478
Itoh N and Tsuneto T 1972 *Prog. Theor. Phys.* **48** 1849
Nambu Y 1960 *Phys. Rev.* **117** 648
Pines D 1971 *Proc. 12th Int. Conf. on Low Temperature Physics, Kyoto Japan* vol 7, ed E Kanda (Tokyo: Academic Press of Japan)
Takatsuka T 1972 *Prog. Theor. Phys.* **48** 1517
Tsuruta S, Canuto V, Lodenquai J and Ruderman M 1972 *Astrophys. J.* **176** 739